

Bounds for Pach's selection theorem and for the minimum solid angle in a simplex

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Abstract

We focus on the estimates on the selection constant in the following geometric selection theorem by Pach: For every positive integer d there is a constant $c_d > 0$ such that whenever X_1, \dots, X_{d+1} are n -element subsets of \mathbb{R}^d , then we can find a point $\mathbf{p} \in \mathbb{R}^d$ and subsets $Y_i \subseteq X_i$ for every $i \in [d+1]$, each of size at least $c_d n$, such that \mathbf{p} belongs to all *rainbow* d -simplices determined by Y_1, \dots, Y_{d+1} , that is, simplices with one vertex in each Y_i .

We show an exponentially decreasing upper bound $c_d \leq \kappa^d$ for a suitable constant $\kappa < 1$. The ideas used in the proof of the upper bound also help us prove Pach's theorem with $c_d > 2^{-2^{d^2+O(d)}}$, which is a lower bound doubly exponentially decreasing in d (up to a polynomial in the exponent). For comparison, Pach's original approach yields a triply exponentially decreasing lower bound. On the other hand, Fox, Pach, and Suk recently announced a hypergraph density result implying a proof of Pach's theorem with $c_d > 2^{-O(d^3 \log d)}$.

In our construction for the upper bound, we use the fact that the minimum solid angle of every d -simplex is exponentially small. This fact was previously unknown and might be of independent interest. For the lower bound, we improve the 'separation' part of the argument by showing that in one of the key steps only $d+1$ separations are necessary, compared to 2^d separations in the original proof.

We also provide a measure version of Pach's theorem.

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1 Introduction

Selection theorems have attracted a lot of interest in discrete geometry. We focus on the positive fraction selection theorem by Pach [Pac98]. For a more compact statement, we introduce the following piece of terminology first. Let S_1, \dots, S_{d+1} be subsets of \mathbb{R}^d . By an (S_1, \dots, S_{d+1}) -simplex we mean the convex hull of points s_1, \dots, s_{d+1} where $s_i \in S_i$ for $i \in [d+1]$. Note that an (S_1, \dots, S_{d+1}) -simplex might be degenerate if the points s_i are not in general position. See Figure 1 illustrating the statement of the theorem.

Theorem 1 (Pach [Pac98]). *For every positive integer d , there exists a constant $c_d > 0$ with the following property. Let X_1, \dots, X_{d+1} be n -element subsets of \mathbb{R}^d . Then there exist a point $\mathbf{p} \in \mathbb{R}^d$ and subsets $Y_i \subseteq X_i$ for $i \in [d+1]$ each of them of size at least $c_d n$ such that the point \mathbf{p} belongs to all (Y_1, \dots, Y_{d+1}) -simplices.*

For a fixed d , we denote by c_d^{\sup} the supremum of the constants with which the theorem remains valid and we call this value *Pach's (selection) constant*.¹ Our aim is to estimate bounds on c_d^{\sup} . Although Pach's proof of Theorem 1 is nice and elegant, it uses several heavy tools: a weaker selection theorem, the weak hypergraph regularity lemma, and the same-type lemma. These tools yield a lower bound on c_d^{\sup} , which is roughly triply exponentially decreasing in d .

We aim at tighter bounds for c_d^{\sup} . We will show an exponentially decreasing upper bound on c_d^{\sup} . The idea for the construction for the upper bound is relatively straightforward. We just place the points of the sets X_1, \dots, X_{d+1} uniformly in the unit ball. The analysis of this construction requires two important ingredients. One ingredient is the analysis of the regions where the sets Y_i from Theorem 1 can appear. Using a certain separation lemma (see Lemma 7) we can deduce that they appear in "corner regions" of arrangements of $d+1$ hyperplanes. The second ingredient is an upper bound on the minimum *solid* angle in a simplex. This bound helps us to bound the sizes of the corner regions for Y_i . We could not find any bound on the minimum solid angle in a simplex in the literature. We provide an exponentially decreasing upper bound, which might be of independent interest.

The description of the corner regions and Lemma 7 also allow us to obtain a doubly exponentially decreasing lower bound on c_d^{\sup} . More concretely, we will show that $c_d^{\sup} \geq 2^{-2^{d^2+O(d)}}$. Shortly before making a preprint version of this paper publicly available, we have learned that Fox, Pach, and Suk expect to obtain an impressive lower bound $c_d^{\sup} \geq 2^{-O(d^3 \log d)}$.

Theorem 2. *Pach's selection constant can be bounded as follows.*

1. $c_d^{\sup} \leq \kappa^d$ where $\kappa < 1$; (our proof will be done with $\kappa = 0.976$) and
2. $c_d^{\sup} \geq 2^{-2^{d^2+3d}}$.

The minimum solid angle of a simplex is discussed in Section 2. Section 3 contains the description of the corner regions and the separation lemma (Lemma 7) we need. Section 4 contains the proof of Theorem 2(1) and Section 5 the proof of Theorem 2(2).

¹Although we are interested in the dependence of c_d^{\sup} on d , we call it a constant emphasizing its independence on the size of the sets X_i .

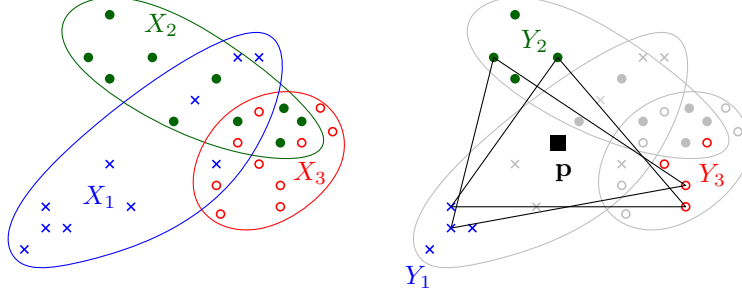


Figure 1: Pach's theorem: initial configuration (left) and the resulting sets Y_i and the resulting point \mathbf{p} (right).

Other selection theorems.

The following weaker selection theorem is related to the positive fraction selection theorem of Pach (the general position assumption is not crucial but we choose the simplest statement in this case).

Theorem 3. *For every $d \in \mathbb{N}$, there is a constant $k_d > 0$ with the following property. Let P be a set of n points in general position in \mathbb{R}^d . Then there is a point in at least $k_d \cdot \binom{n}{d+1} - O(n^d)$ d -simplices spanned by P .*

Note that $\binom{n}{d+1}$ is the number of all d -simplices spanned by P , thus the statement of Theorem 3 says that we can indeed select a positive fraction of simplices sharing a point. It is not hard to see that Theorem 3 follows from Theorem 1 as soon as only the existence of k_d is concerned (by splitting P into X_1, \dots, X_{d+1} , possibly forgetting few points).

The planar case of Theorem 3 is due to Boros and Füredi [BF84] ($d = 2$); it was extended to arbitrary dimension by Bárány [Bár82]. Bárány proved the theorem with $k_d = \frac{1}{(d+1)^d}$.

A significant improvement to k_d was found by Gromov [Gro10] by topological method in quite more general setting (obtaining a proof with $k_d = \frac{1}{(d+1)!}$). Karasev [Kar12] found a simpler proof (still in quite general setting) and Matoušek and Wagner [MW11] extracted the combinatorial essence of Gromov's proof allowing them to get a further (slight) improvement on k_d . Král', Mach and Sereni [KMS12] obtained a further improvement of the value focusing on the combinatorial part extracted by Matoušek and Wagner. We do not attempt to enumerate the bounds obtained in [MW11, KMS12].

The following variant of Theorem 3 for rainbow simplices is an important step in the proof of Theorem 1.

Theorem 4. *For every $d \in \mathbb{N}$, there is a constant $k'_d > 0$ with the following property. Let X_1, \dots, X_{d+1} be pairwise disjoint n -element subsets of \mathbb{R}^d whose union is in general position. Then there is a point $\mathbf{p} \in \mathbb{R}^d$ which is contained in the interior of at least $k'_d \cdot n^{d+1} - O(n^d)$ rainbow d -simplices, where a rainbow simplex meets each X_i in exactly one vertex and $k'_d > 0$ is a constant depending only on d .*

Theorem 4 is implicitly proved in [Pac98] with k'_d roughly around $\frac{1}{(5d)^{d^2}}$. Karasev's proof [Kar12] (following Gromov) gives the result with $k'_d = \frac{1}{(d+1)!}$. Note that Karasev's main result is in the setting of absolutely continuous measures. It can be easily transformed

into the setting of Theorem 4 by replacing each point $x \in X_1 \cup \dots$ by a sufficiently small ball centered in x and using the fact that any point of \mathbb{R}^d can be at the boundary of at most $O(n^d)$ simplices spanned by $X_1 \cup \dots \cup X_{d+1}$; see [Mat02, Lemma 9.1.2].²

An interesting selection theorem in a ‘kind of’ dual setting was recently obtained by Bárány and Pach [BP14]. A variant of Pach’s theorem for hypergraphs with bounded degree was, also recently, obtained by Fox et al. [FGL⁺12].

Measure version of Pach’s theorem.

Due to the similarity of Pach’s theorem to other geometric selection theorems, such as Theorem 4, we can expect that Pach’s theorem also admits a measure version where point sets are replaced with probability measures. We will indeed verify this expectation (with the same value for the selection constant).

We say that a measure μ on \mathbb{R}^d is *absolutely continuous* if $\mu(A) = 0$ whenever the Lebesgue measure of A is 0. Implicitly in the definition we assume that the σ -algebra of μ -measurable sets coincides with the σ -algebra of Lebesgue-measurable sets.

Theorem 5. *Let μ_1, \dots, μ_{d+1} be absolutely continuous probability measures in \mathbb{R}^d . Then there exist sets $Z_i \subseteq \mathbb{R}^d$ with $\mu(Z_i) \geq 2^{-2^{d^2+3d}}$ and a point $\mathbf{p} \in \mathbb{R}^d$ which is contained in all (Z_1, \dots, Z_{d+1}) -simplices.*

Our proof of Theorem 5 closely follows the proof of the discrete version, Theorem 2(2). We need to replace several tools used in the proof of the discrete version by their measure-theoretic analogues. One of these tools, the weak hypergraph regularity lemma, does not seem to be established in the literature in the setting we need. In this case, it is still possible to obtain the measure-theoretic version of the weak hypergraph regularity lemma by modifying the proof for the discrete version.

We believe that establishing Theorem 5 is interesting for completeness. In addition, some steps in the proof of Theorem 5 are actually even easier than their analogues in the discrete version, since careful considerations of general position in the proof of Theorem 2(2) can be left out. However, due to the similarity of many steps in the proofs in these two settings and also due to the fact that a measure-theoretic version of the weak hypergraph regularity lemma is quite technical, we leave the proof of Theorem 5 to the appendix.

2 The minimum solid angle in a simplex

We start our preparations for the proof of Theorem 2(1) by bounding the minimum solid angle in a simplex.

Let Δ be a d -simplex and v be a vertex of Δ . By the *solid angle* at \mathbf{v} in Δ we mean the value

$$\text{sa}(\mathbf{v}; \Delta) := \frac{\text{Vol}(B(\mathbf{v}; \varepsilon) \cap \Delta)}{\text{Vol}(B(\mathbf{v}; \varepsilon))}$$

where $B(\mathbf{x}; r)$ denotes the ball centered in \mathbf{x} with radius r ; ε is small enough (so that $B(\mathbf{v}; \varepsilon)$ does not meet the hyperplane determined by the vertices of Δ except \mathbf{v}); and Vol denotes

²More precisely, Lemma 9.1.2 in [Mat02] also implies that for sufficiently small ε (depending on X_1, \dots, X_{d+1}), any point of \mathbb{R}^d can be ε -close to the boundary of at most $O(n^d)$ simplices. Indeed, thickening the boundaries by small enough $\varepsilon > 0$ does not introduce new intersections.

Now, it is sufficient to choose the radii of the small balls around points of X_1, \dots, X_{d+1} equal to ε .

the d -dimensional volume (that is, the d -dimensional Lebesgue measure). Note that in our case the solid angle is normalized, that is, it measures the probability that a random point of $B(\mathbf{v}; \varepsilon)$ belongs to the simplex.

Our goal is to give the upper bound on the minimum solid angle of Δ :

$$\text{msa}(\Delta) := \min\{\text{sa}(\mathbf{v}; \Delta) : \mathbf{v} \text{ is a vertex of } \Delta\}.$$

Theorem 6. *The minimum solid angle of an arbitrary d -simplex Δ satisfies $\text{msa}(\Delta) \leq \gamma^d$ where $\gamma < 1$ is independent of d .*

We prove the theorem with $\gamma = \sqrt{\frac{\sqrt{3}}{2}} \leq 0.931$.

Proof. Let \mathbf{ab} be one of the longest edges of Δ . Without loss of generality, at least half of the remaining vertices of Δ are not farther from \mathbf{b} than from \mathbf{a} . Let $\mathbf{v}_1, \dots, \mathbf{v}_k$ be such vertices ($k \geq (d-1)/2$) and let $\mathbf{u}_1, \dots, \mathbf{u}_\ell$ be the remaining vertices, which are closer to \mathbf{a} than to \mathbf{b} . We observe that the angles $\mathbf{v}_i \mathbf{ab}$ are at most 60° since $\mathbf{v}_i \mathbf{b}$ is one of the shortest edges in the triangle \mathbf{abv}_i . We also observe that the angles $\mathbf{u}_i \mathbf{ab}$ are at most 90° since \mathbf{bu}_i is at most as long as \mathbf{ab} .

Let h be the hyperplane perpendicular to \mathbf{ab} passing through \mathbf{a} and let h^+ be the closed halfspace bounded by the hyperplane h and containing \mathbf{b} . Let C be the cone with apex \mathbf{a} determined by Δ . Fix a sufficiently small $\varepsilon > 0$ such that the ball $B(\mathbf{a}, \varepsilon)$ with center \mathbf{a} and radius ε does not meet the hyperplane determined by the vertices of Δ other than \mathbf{v} . We need to determine what fraction of the ball $B(\mathbf{a}, \varepsilon)$ belongs to C . Since all the angles $\mathbf{v}_i \mathbf{ab}$ and $\mathbf{u}_i \mathbf{ab}$ are at most 90° , it follows that C is fully contained in h^+ .

Let κ be the affine $(k+1)$ -space determined by $\mathbf{a}, \mathbf{b}, \mathbf{v}_1, \dots, \mathbf{v}_k$. Let C_{60}^κ be the $(k+1)$ -dimensional cone formed by all points \mathbf{x} in κ such that the angle \mathbf{xab} is at most 60° . From the discussion above it follows that C_{60}^κ contains all the vertices \mathbf{v}_i , and consequently $C \cap \kappa \subseteq C_{60}^\kappa$. It is not too difficult to show that

$$\frac{\text{Vol}_{k+1}(B(\mathbf{a}, \varepsilon) \cap C_{60}^\kappa)}{\text{Vol}_{k+1}(B(\mathbf{a}, \varepsilon) \cap \kappa)} \leq \left(\frac{\sqrt{3}}{2}\right)^{k+1}, \quad (1)$$

where Vol_{k+1} is the $(k+1)$ -dimensional volume in κ . Indeed, the set $B(\mathbf{a}, \varepsilon) \cap C_{60}^\kappa$ is contained in the $(k+1)$ -dimensional ball inside κ of radius $\frac{\sqrt{3}}{2}\varepsilon$; see the dashed circle in Figure 2, left. (We have borrowed this idea from [R  c06], aiming at a reasonable estimate without precise computation.)

Now we estimate $\text{Vol}(B(\mathbf{a}, \varepsilon) \cap C) / \text{Vol}(B(\mathbf{a}, \varepsilon))$. Let κ' be an arbitrary $(k+1)$ -space parallel to κ . Our goal is to show that

$$\frac{\text{Vol}_{k+1}(B(\mathbf{a}, \varepsilon) \cap C \cap \kappa')}{\text{Vol}_{k+1}(B(\mathbf{a}, \varepsilon) \cap \kappa')} \leq \left(\frac{\sqrt{3}}{2}\right)^{k+1}. \quad (2)$$

As soon as we show (2) we get the same bound on $\text{Vol}(B(\mathbf{a}, \varepsilon) \cap C) / \text{Vol}(B(\mathbf{a}, \varepsilon))$ by the Fubini theorem.

In order to show (2), we first observe that $C \cap \kappa'$ is either empty or it equals $(C \cap \kappa) + \mathbf{y}$, where \mathbf{y} is the intersection point of κ' and the ℓ -dimensional cone with apex \mathbf{a} determined by the points $\mathbf{u}_1, \dots, \mathbf{u}_\ell$ (here, for simplicity, we assume that \mathbf{a} coincides with the origin). Thus, in particular, $\mathbf{y} \in h^+$ and $C \cap \kappa' \subseteq \mathbf{y} + C_{60}^\kappa$. See Figure 2, right.

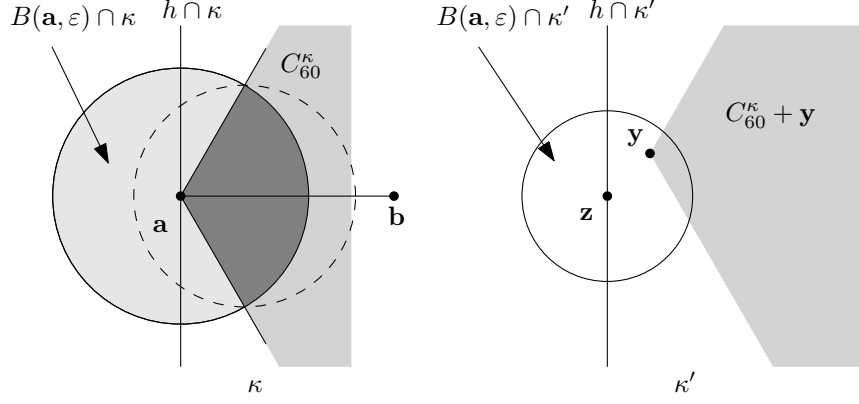


Figure 2: Cuts through κ and κ' .

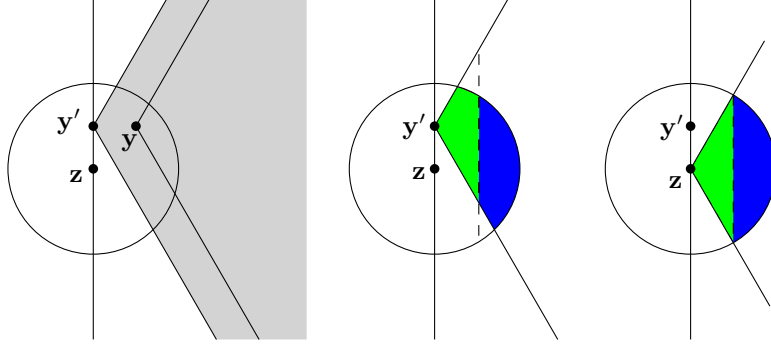


Figure 3: Bounding the volume of $B(\mathbf{a}, \varepsilon) \cap (\mathbf{y} + C_{60}^\kappa)$.

The next step is to show that $\text{Vol}_{k+1}(B(\mathbf{a}, \varepsilon) \cap (\mathbf{y} + C_{60}^\kappa)) \leq \text{Vol}_{k+1}(B(\mathbf{a}, \varepsilon) \cap (\mathbf{z} + C_{60}^\kappa))$ where \mathbf{z} is the center of the $(k+1)$ -dimensional ball $B(\mathbf{a}, \varepsilon) \cap \kappa'$. See Figure 3, left. We first shift \mathbf{y} to its orthogonal projection \mathbf{y}' on h , observing that $\text{Vol}_{k+1}(B(\mathbf{a}, \varepsilon) \cap (\mathbf{y} + C_{60}^\kappa)) \leq \text{Vol}_{k+1}(B(\mathbf{a}, \varepsilon) \cap (\mathbf{y}' + C_{60}^\kappa))$. Then we bound $\text{Vol}_{k+1}(B(\mathbf{a}, \varepsilon) \cap (\mathbf{y}' + C_{60}^\kappa))$ by decomposing $B(\mathbf{a}, \varepsilon) \cap (\mathbf{y}' + C_{60}^\kappa)$ into two parts as in the middle and the right part of Figure 3.

Analogously to (1), we have $\text{Vol}_{k+1}(B(\mathbf{a}, \varepsilon) \cap (\mathbf{z} + C_{60}^\kappa)) / \text{Vol}_{k+1}(B(\mathbf{a}, \varepsilon) \cap \kappa') \leq \left(\frac{\sqrt{3}}{2}\right)^{k+1}$. Hence, we deduce (2). This gives the final bound

$$\text{msa}(\Delta) \leq \left(\frac{\sqrt{3}}{2}\right)^{(d+1)/2} \leq (0.931)^d.$$

□

3 Corner selection

In this section we describe a geometric structure we are essentially looking for in order to prove Theorem 2.

General position assumptions.

Although Theorem 1 does not assume any kind of general position, we will need general position in our intermediate steps. Thus we start with a remark on the general position assumptions we are going to use.

Given a set X of points in \mathbb{R}^d we say that it is in *general position* if it satisfies the following condition.

- (G) Whenever X_1, \dots, X_k are pairwise disjoint subsets of X , then $\text{codim}(\text{aff}(X_1) \cap \dots \cap \text{aff}(X_k)) = \min\{d+1, \text{codim}(\text{aff}(X_1)) + \dots + \text{codim}(\text{aff}(X_k))\}$ where $\text{codim } A$ denotes the codimension of an affine space A , that is, $d - \dim A$. By convention $\text{codim } \emptyset = d+1$.

Note that condition (G) implies that any subset of X of size at most $d+1$ is affinely independent. Given a set X' which is not in general position, there is arbitrarily small perturbation of points in X' yielding a set satisfying (G). In most of the cases we will only need the affine independence. The condition (G) might seem technical but it generalizes the following situation in the plane. Let $\ell_1 = \mathbf{a}_1\mathbf{b}_1$, $\ell_2 = \mathbf{a}_2\mathbf{b}_2$, and $\ell_3 = \mathbf{a}_3\mathbf{b}_3$ be three lines in the plane determined by six distinct points of X . Then, in general position, we might expect that these three lines do not meet in a point.

We will also work with arrangements of $d+1$ hyperplanes. We say that such an arrangement is in *general position* if the normal vectors of arbitrary d hyperplanes from the arrangement are linearly independent (in particular each d of the hyperplanes have a single point in common) and if the intersection of all $d+1$ of the hyperplanes is empty.

Corner regions.

Let $\mathcal{H} = (H_1, \dots, H_{d+1})$ be an arrangement of hyperplanes in \mathbb{R}^d in general position. For $i \in [d+1]$ let \mathbf{h}_i denote the intersection point of all hyperplanes from \mathcal{H} but H_i . It is not difficult to see that the arrangement \mathcal{H} has exactly one bounded component, namely the simplex with vertices \mathbf{h}_i . We denote this simplex by $\Delta(\mathcal{H})$. We also denote by H_i^+ and H_i^- the two closed subspaces determined by H_i in such a way that H_i^- contains $\Delta(\mathcal{H})$. Finally, we define the *corner regions* $C_i = C_i(\mathcal{H})$ by setting

$$C_i := \bigcap_{j \in [d+1] \setminus \{i\}} H_j^+.$$

Note that each C_i is a cone with apex \mathbf{h}_i ; see Figure 4, left.

For the success of our approach we need the following separation lemma. Given a collection (S_1, \dots, S_k) of sets, by \widehat{S}_i we mean the set $S_1 \cup \dots \cup S_{i-1} \cup S_{i+1} \cup \dots \cup S_k$, for any $i \in [k]$. The interior of a set $S \subseteq \mathbb{R}^d$ is denoted by $\text{int } S$.

Lemma 7. *Let \mathbf{p} be a point in \mathbb{R}^d , \mathcal{H} be an arrangement of $(d+1)$ hyperplanes in general position in \mathbb{R}^d and Y_1, \dots, Y_{d+1} be subsets of \mathbb{R}^d (not necessarily finite this time) such that H_i strictly separates \mathbf{p} from \widehat{Y}_i for every $i \in [d+1]$ (in particular \mathbf{p} does not belong to any H_i). Then either*

- $\mathbf{p} \in \Delta(\mathcal{H})$ and $Y_i \subseteq \text{int } C_i$ for any $i \in [d+1]$; or
- $\mathbf{p} \notin \Delta(\mathcal{H})$ and there is a hyperplane strictly separating \mathbf{p} from $Y_1 \cup \dots \cup Y_{d+1}$ (see Figure 5).

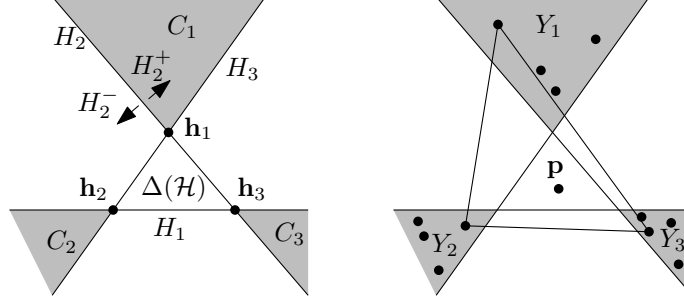


Figure 4: Corner regions of an arrangement of $d + 1$ hyperplanes.

In the proof of Lemma 7 we need the following lemma on rescaling the normal vectors of a simplex. See, for example, [Kla04, Proposition 1] and the references therein.

Lemma 8. *Let $\mathcal{H} = (H_1, \dots, H_{d+1})$ be an arrangement of $(d + 1)$ hyperplanes in general position and let $\mathbf{u}_1, \dots, \mathbf{u}_{d+1}$ be the unit normal vectors to H_1, \dots, H_{d+1} pointing outwards of $\Delta(\mathcal{H})$. Then there are positive coefficients $\alpha_1, \dots, \alpha_{d+1}$ such that*

$$\alpha_1 \mathbf{u}_1 + \dots + \alpha_{d+1} \mathbf{u}_{d+1} = \mathbf{0}.$$

Proof of Lemma 7. For each $i \in [d + 1]$ the point \mathbf{p} either belongs to $\text{int } H_i^+$ or $\text{int } H_i^-$. Let $I \subseteq [d + 1]$ be the set of those $i \in [d + 1]$ for which \mathbf{p} belongs to $\text{int } H_i^+$.

Note that $I \neq [d + 1]$, since otherwise $\mathbf{p} \in \emptyset$. If I is empty, then \mathbf{p} belongs to $\Delta(\mathcal{H})$ according to our definitions. Given $i, j \in [d + 1]$ such that $i \neq j$ we get $Y_i \subseteq \text{int } H_j^+$ since H_j separates \mathbf{p} and \hat{Y}_j which contains Y_i . This implies $Y_i \subseteq \text{int } C_i$ by considering the previous inclusion for all $j \neq i$.

If I is nonempty, then $\mathbf{p} \notin \Delta(\mathcal{H})$ and our task is to separate \mathbf{p} from $Y_1 \cup \dots \cup Y_{d+1}$.

Without loss of generality, let us assume that the origin $\mathbf{0}$ belongs to $\text{int}(\Delta(\mathcal{H}))$. Let $\mathbf{n}_i := \alpha_i \mathbf{u}_i$ be the normal vectors to the hyperplanes where α_i and \mathbf{u}_i come from Lemma 8, in particular $\mathbf{n}_1 + \dots + \mathbf{n}_{d+1} = \mathbf{0}$. Then the hyperplane H_i is given by the equation $H_i = \{\mathbf{x} \in \mathbb{R}^d : \mathbf{x} \cdot \mathbf{n}_i = c_i\}$ where $c_i > 0$ and the corresponding halfspaces satisfy $H_i^+ = \{\mathbf{x} \in \mathbb{R}^d : \mathbf{x} \cdot \mathbf{n}_i \geq c_i\}$ and $H_i^- = \{\mathbf{x} \in \mathbb{R}^d : \mathbf{x} \cdot \mathbf{n}_i \leq c_i\}$. We further set $\mathbf{n} := \sum_{i \in I} \mathbf{n}_i$, $c := \sum_{i \in I} c_i$ and we consider the hyperplane $H := \{\mathbf{x} \in \mathbb{R}^d : \mathbf{x} \cdot \mathbf{n} = c\}$. The task is to show that H is the desired separating hyperplane.

We first observe that $\mathbf{p} \cdot \mathbf{n} = \sum_{i \in I} \mathbf{p} \cdot \mathbf{n}_i > \sum_{i \in I} c_i = c$ (the inequality holds due to the definition of I), in particular $\mathbf{n} \neq \mathbf{0}$. Thus our task reduces to showing that each Y_i belongs to the interior of the half space $H^- := \{\mathbf{x} \in \mathbb{R}^d : \mathbf{x} \cdot \mathbf{n} \leq c\}$. We distinguish two cases according to whether $i \in I$.

1. In the first case we assume $i \in I$. Let us consider $\mathbf{y} \in Y_i$ aiming to show $\mathbf{y} \in \text{int } H^-$. Let $j \in [d + 1] \setminus I$, in particular $j \neq i$. Since H_j separates \mathbf{p} and \hat{Y}_j we get that it, in particular, separates \mathbf{p} and \mathbf{y} . Thus, $\mathbf{y} \cdot \mathbf{n}_j > c_j$ by the definition of I . Summing up we get

$$\mathbf{y} \cdot \sum_{j \in [d+1] \setminus I} \mathbf{n}_j > \sum_{j \in [d+1] \setminus I} c_j > -c.$$

The left-hand side of the equation above, however, equals $-\mathbf{y} \cdot \mathbf{n}$ since $\sum_{k \in [d+1]} \mathbf{n}_k = \mathbf{0}$. This yields the desired conclusion $\mathbf{y} \in \text{int } H^-$ (by multiplying by -1).

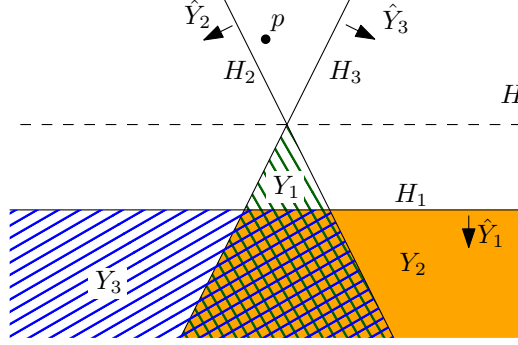


Figure 5: Illustration for Lemma 7. The point \mathbf{p} belongs to $H_1^- \cap H_2^+ \cap H_3^+$ in this case; $Y_1 \subseteq \hat{Y}_2 \cap \hat{Y}_3$ and so on. The shaded regions denote where the sets Y_1 , Y_2 and Y_3 may appear.

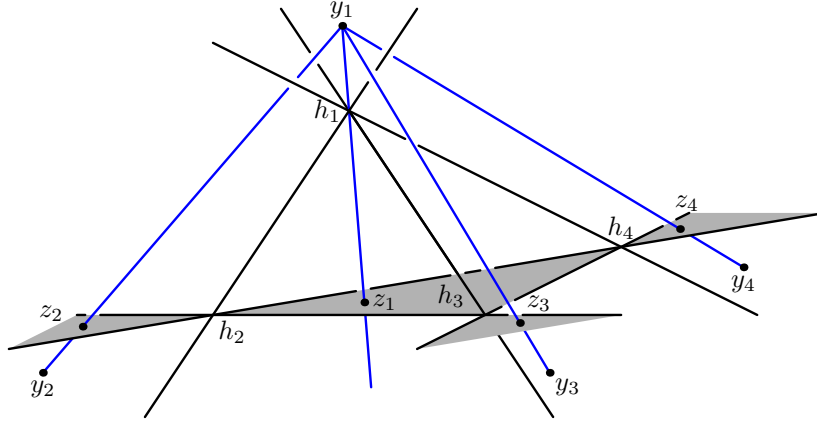


Figure 6: The case $d = 3$

2. In the second case we assume $i \in [d+1] \setminus I$. We again consider $\mathbf{y} \in Y_i$ aiming to show $\mathbf{y} \in \text{int } H^-$. This time we consider $j \in I$ so that we know $j \neq i$. We again get that H_j separates \mathbf{p} and \mathbf{y} . Therefore $\mathbf{y} \cdot \mathbf{n}_j < c_j$. Summing up over all $j \in i$ we get the desired conclusion $\mathbf{y} \cdot \mathbf{n} < c$.

□

For the proof of Theorem 2(2) we need to verify, an intuitively obvious fact, that the corner selection yields the Pach point. This part is not needed for Theorem 2(1).

Lemma 9. *Let $\mathcal{H} = (H_1, \dots, H_{d+1})$ be an arrangement of $(d+1)$ hyperplanes in general position. Let \mathbf{p} be a point in $\Delta(\mathcal{H})$ and $\mathbf{y}_1, \dots, \mathbf{y}_{d+1}$ be points in \mathbb{R}^d such that $\mathbf{y}_i \in C_i$ for any $i \in [d+1]$. Then \mathbf{p} belongs to the simplex determined by $\mathbf{y}_1, \dots, \mathbf{y}_{d+1}$.*

Proof. We prove the lemma by induction on d . For $d = 1$ is the proof obvious; therefore we can focus on the second induction step assuming $d > 1$.

We recall that each C_i is a cone with apex \mathbf{h}_i . Since \mathbf{p} is a convex combination of the points \mathbf{h}_i , it is sufficient to show that each \mathbf{h}_i belongs to the simplex determined by $\mathbf{y}_1, \dots, \mathbf{y}_{d+1}$.

We fix i and consider points $\mathbf{z}_i := \overline{\mathbf{h}_i \mathbf{y}_i} \cap H_i$ and $\mathbf{z}_j := \overline{\mathbf{y}_i \mathbf{y}_j} \cap H_i$ for $j \in [d+1] \setminus \{i\}$, where $\overline{\mathbf{ab}}$ is the line spanned by points \mathbf{a} and \mathbf{b} . See Figure 6.

We claim that \mathbf{z}_j , for $j \neq i$, belongs to C'_j , where $C'_j := C_j \cap H_i$ is the corner region with apex \mathbf{h}_j in $H_i \simeq \mathbb{R}^{d-1}$. Indeed, since $\mathbf{y}_j \in C_j = \bigcap_{k \in [d+1] \setminus \{j\}} H_k^+$ we have $\overline{\mathbf{y}_i \mathbf{y}_j} \in \bigcap_{k \in [d+1] \setminus \{i,j\}} H_k^+$, thus $\mathbf{z}_j \in H_i \cap (\bigcap_{k \in [d] \setminus \{j\}} H_k^+)$, what is by definition the corner C'_j . We also observe that $\mathbf{z}_i \in \Delta(\mathcal{H}')$, where $\mathcal{H}' = (\mathcal{H} \setminus \{H_i\}) \cap H_i$ is an arrangement of d hyperplanes in $H_i \simeq \mathbb{R}^{d-1}$. Therefore, by induction, \mathbf{z}_i is in the convex hull of points \mathbf{z}_j (for $j \in [d+1] \setminus \{i\}$). Since all these points \mathbf{z}_j are by definition convex combination of points $\mathbf{y}_1, \dots, \mathbf{y}_{d+1}$ and since \mathbf{h}_i is a convex combination of \mathbf{y}_i and \mathbf{z}_i we deduce that \mathbf{h}_i is in the simplex determined by $\mathbf{y}_1, \dots, \mathbf{y}_{d+1}$ as we require. \square

4 Upper bound

The task of this section is to give an exponentially decreasing upper bound on c_d^{\sup} . As we sketched in the introduction, we set X_i , for $i \in [d+1]$ to be sets of n points uniformly distributed in the unit d -ball B^d . We will explain later what do we exactly mean with a uniform distribution. The idea is that if A is ‘sufficiently nice’ subset of B^d then $\frac{\text{Vol}(A)}{\text{Vol}(B^d)}$ is approximately equal to $\frac{|X_i \cap A|}{|X_i|}$.

To avoid many repetitions, it is convenient to introduce the following terminology. By a *generic Pach’s configuration* we mean a collection $(Y_1, \dots, Y_{d+1}, \mathbf{p})$ of $d+1$ finite pairwise disjoint nonempty sets Y_i and a point \mathbf{p} not belonging to any Y_i such that the set $Y_1 \cup \dots \cup Y_{d+1} \cup \{\mathbf{p}\}$ is in general position and \mathbf{p} belongs to all (Y_1, \dots, Y_{d+1}) -simplices.

Note that if we consider $(Y_1, \dots, Y_{d+1}, \mathbf{p})$ as output of Theorem 1 we need not obtain generic Pach’s configuration even if $X := X_1 \cup \dots \cup X_{d+1}$ is in general position (the point \mathbf{p} might be on some of the hyperplanes determined by X). In such case, forgetting few points only, we still can get a generic Pach’s configuration.

Lemma 10. *Let Y'_1, \dots, Y'_{d+1} be $d+1$ finite pairwise disjoint sets of size at least $d+1$ such that $Y'_1 \cup \dots \cup Y'_{d+1}$ is in general position. Let \mathbf{p}' be a point contained in all (Y'_1, \dots, Y'_{d+1}) -simplices. Then there are subsets $Y_i \subseteq Y'_i$ for $i \in [d+1]$ such that $|Y_i| \geq |Y'_i| - d$, and a point $\mathbf{p} \in \mathbb{R}^d$ such that $(Y_1, \dots, Y_{d+1}, \mathbf{p})$ is a generic Pach’s configuration.*

Proof. Let $\Delta_1, \dots, \Delta_k$ be a maximal collection of (Y'_1, \dots, Y'_{d+1}) -simplices such that \mathbf{p}' is on the boundary of each Δ_i for $i \in [k]$ and any two simplices of this collection have disjoint vertex sets.

Let F_i be a proper face of Δ_i containing \mathbf{p}' . By the general position assumption (G), introduced in Section 3 applied to $\text{aff}(F_1), \dots, \text{aff}(F_k)$ we deduce that $k \leq d$ since each $\text{aff}(F_i)$ has codimension at least 1 and all of them intersect in an affine space containing \mathbf{p}' (possibly only \mathbf{p}').

Now, we remove vertices of $\Delta_1, \dots, \Delta_k$ from each Y'_i , obtaining sets Y_i , removing at most d points from each Y'_i . Then \mathbf{p}' is in the interior of all (Y_1, \dots, Y_{d+1}) -simplices due to the maximality of $\Delta_1, \dots, \Delta_k$. By a small perturbation of \mathbf{p}' we get a point \mathbf{p} still in the interior of all (Y_1, \dots, Y_{d+1}) -simplices. Then $(Y_1, \dots, Y_{d+1}, \mathbf{p})$ is the required generic Pach’s configuration. \square

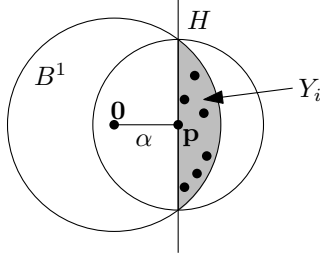


Figure 7: Cap fits into a smaller ball.

We also need a proposition saying that if $(Y_1, \dots, Y_{d+1}, \mathbf{p})$ is a generic Pach's configuration in the unit ball B^d (centered in the origin $\mathbf{0}$), then some Y_i is in a tiny part of the ball (depending on the distance of \mathbf{p} from the center of the ball). By β_d we denote the volume of the unite d -ball.

Proposition 11. *Let $(Y_1, \dots, Y_{d+1}, \mathbf{p})$ be a generic Pach's configuration such that $Y_1 \cup \dots \cup Y_{d+1} \cup \{\mathbf{p}\}$ is subset of B^d . Let α be the distance of \mathbf{p} and $\mathbf{0}$. Then*

1. *there is $\ell \in [d+1]$ such that Y_ℓ is contained in a solid cap cut out of B^d of volume at most $(1 - \alpha)^{d/2} \beta_d$; and*
2. *there is an arrangement of hyperplanes $\mathcal{H} = (H_1, \dots, H_{d+1})$ in general position such that each Y_i belongs to the corner region $C_i = C_i(\mathcal{H})$ (see the definitions in Section 3). The smallest of the volumes $\text{Vol}(C_i \cap B^d)$ is at most $(1 + \alpha)^d \text{msa}(\Delta(\mathcal{H})) \beta_d$ (recalling that msa denotes the minimum solid angle).*

For a proof we need the following simple lemma.

Lemma 12. *Let $(Y_1, \dots, Y_{d+1}, \mathbf{p})$ be a generic Pach's configuration. Let H be any hyperplane passing through \mathbf{p} . Then for any of the two open halfspaces determined by H there is $\ell \in [d+1]$ such that Y_ℓ is fully contained in that halfspace.*

Proof. For contradiction let us assume that each Y_i meet the opposite closed halfspace in a point \mathbf{y}_i . Since \mathbf{p} belongs to the simplex formed by these \mathbf{y}_i , it belongs to the convex hull of those \mathbf{y}_i which are in H . This contradicts the general position assumption from generic Pach's configuration. \square

Proof of Proposition 11. We first prove the first part of the proposition.

Consider the hyperplane H passing through \mathbf{p} perpendicular to the line $\mathbf{0p}$. By Lemma 12 there is ℓ such that Y_ℓ is in the solid cap cut by H . The volume of this solid cap is at most $(1 - \alpha)^{d/2}$ since it fits into a ball of radius $\sqrt{1 - \alpha}$; see Figure 7.

Now we can prove the second item in the proposition.

As a first step, we need to show an existence of hyperplanes H_i such that each H_i strictly separates \mathbf{p} from \hat{Y}_i (recalling the notation $\hat{Y}_i = \bigcup_{j \in [d+1] \setminus \{i\}} Y_j$).

For contradiction let us assume that for some $i \in [d+1]$ the point \mathbf{p} is not strictly separated from \hat{Y}_i by a hyperplane. That means that \mathbf{p} belongs to the convex hull $\text{conv}(\hat{Y}_i)$. Consequently, there are points $\mathbf{z}_j \in \text{conv}(Y_j)$ for $j \in [d+1] \setminus \{i\}$ such that \mathbf{p} is a convex

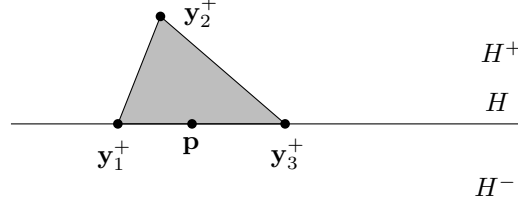


Figure 8: The point \mathbf{p} cannot be in the interior of Δ .

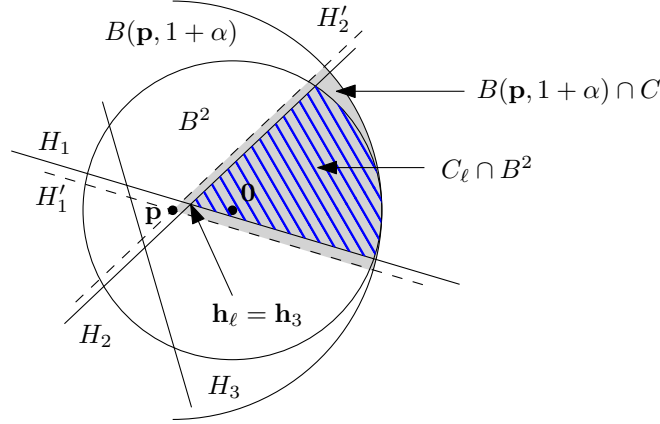


Figure 9: $C \cap B(\mathbf{p}, 1 + \alpha)$ contains $C_\ell \cap B^d$.

combination of them. (Indeed, consider \mathbf{p} as a convex combination of points from \widehat{Y}_i and put together points of each Y_j with appropriate weights.)

Let H be a hyperplane passing through the points \mathbf{z}_j for $j \in [d+1] \setminus \{i\}$. In particular, \mathbf{p} belongs to H . Let \mathbf{y}_i^+ be a point of Y_i and let H^+ and H^- be the closed halfspaces determined by H chosen in such a way that $\mathbf{y}_i^+ \in H^+$. For each $j \in [d+1] \setminus \{i\}$ we can find points \mathbf{y}_j^+ in $H^+ \cup Y_j$ since $\text{conv}(Y_j) \cap H \neq \emptyset$. Let $\Delta \subseteq H^+$ be the (Y_1, \dots, Y_{d+1}) -simplex with vertices \mathbf{y}_j^+ for $j \in [d+1]$; see Figure 8. Since $\Delta \subseteq H^+$ and since \mathbf{p} belongs to H , \mathbf{p} cannot be in the interior of (non-degenerate) Δ . This contradicts our genericity assumption.

Therefore we proved the existence of the required hyperplane arrangement $\mathcal{H} = (H_1, \dots, H_{d+1})$. Then Lemma 7 implies that $\mathbf{p} \in \Delta(\mathcal{H})$ and each Y_i belongs to the corner region $C_i(\mathcal{H})$. (The second option of Lemma 7 cannot occur since \mathbf{p} is in all (Y_1, \dots, Y_{d+1}) -simplices.)

It remains to bound the smallest of the volumes $\text{Vol}(C_i \cap B^d)$. See Figure 9 while following the proof. We fix $\ell \in [d+1]$ such that solid angle ϑ at vertex \mathbf{h}_ℓ is the minimum among solid angles of $\Delta(\mathcal{H})$ (we use the same notation for vertices of $\Delta(\mathcal{H})$ as in Section 3). Let H'_i be a hyperplane parallel to H_i passing through \mathbf{p} for $i \in [d+1] \setminus \ell$ and let C be the cell of the arrangement of hyperplanes $(H'_i)_{i \in [d+1] \setminus \ell}$ which contains \mathbf{h}_ℓ . Then C contains C_ℓ and moreover, since the distance of the origin and \mathbf{p} is α , we deduce that $C \cap B(\mathbf{p}, 1 + \alpha)$ contains $C_\ell \cap B^d$. The volume of $C \cap B(\mathbf{p}, 1 + \alpha)$ is $(1 + \alpha)^d \vartheta = (1 + \alpha)^d \text{msa}(\Delta(\mathcal{H})) \beta_d$. This gives the required bound. \square

Now we have all tools to prove the upper bound.

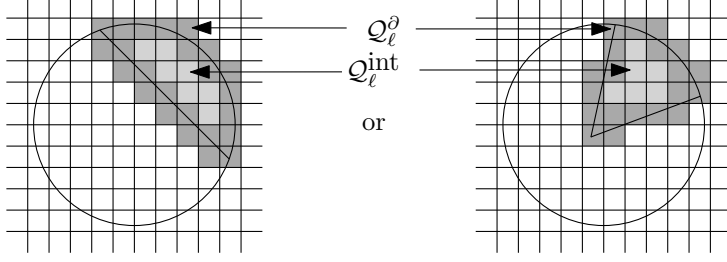


Figure 10: Splitting \mathcal{Q}_ℓ into $\mathcal{Q}_\ell^\partial \cup \mathcal{Q}_\ell^{\text{int}}$. With decreasing ε , the volume of $\bigcup \mathcal{Q}_\ell^\partial$ tends to 0.

Proof of Theorem 2(1). Let $\kappa = 0.976$ and for contradiction let us assume that Theorem 1 is valid with $c_d > \kappa^d$, that is with $c_d = \kappa^d + \zeta$ where $\zeta > 0$.

We consider small $\varepsilon > 0$ and tile \mathbb{R}^d with grid of hypercubes with sides ε . Let \mathcal{Q} be the set of those hypercubes which intersect with B^d in their interiors. For every $Q \in \mathcal{Q}$ and every $i \in [d+1]$ we add exactly one point into X_i belonging to $\text{int } Q \cap \text{int } B^d$. Apart from this we assume that $X_1 \cup \dots \cup X_{d+1}$ is a set in general position. This finishes the construction of the sets X_i .

As usual, n denotes the size of the sets X_i and we observe that it is well approximated by the volume of B^d in the following sense

$$\frac{1}{\varepsilon^d} \beta_d \leq n \leq \frac{1}{\varepsilon^d} (1 + \varepsilon \sqrt{d})^d \beta_d \quad (3)$$

where we recall that β_d is the volume of B^d . (Note that $\bigcup \mathcal{Q}$ fits into a ball of radius $(1 + \varepsilon \sqrt{d})$.)

Since we assume that Theorem 1 is valid with $c_d = \kappa^d + \zeta$ we can use it for our particular X_i and we can get output sets Y'_1, \dots, Y'_{d+1} and a point \mathbf{p}' . Assuming that ε is small enough, that is, n is large enough and using Lemma 10 we can get a generic Pach's configuration $(Y_1, \dots, Y_{d+1}, \mathbf{p})$ where $Y_i \subseteq X_i$ and $|Y_i| > (\kappa^d + \frac{\zeta}{2})|X_i|$ for $i \in [d+1]$.

Let α be the distance of \mathbf{p} and the origin. Obviously, $\alpha < 1$ since each Y_i belong to B^1 .

If $\alpha \geq 0.048$, then there is an $\ell \in [d+1]$ such that Y_ℓ is contained in a solid cap G of volume at most $0.9758^d \beta_d < \kappa^d \beta_d$ by Proposition 11(1).

If $\alpha \leq 0.048$, then there is an $\ell \in [d+1]$ such that Y_ℓ is in the intersection G of a corner region C_ℓ and B^1 such that the volume of G is at most $1.048^d \cdot 0.931^d \beta_d < 0.9757^d \beta_d < \kappa^d \beta_d$ by Proposition 11(2) and Theorem 6.

In both cases, we want bound the number of points Y_ℓ by volume of G . Let \mathcal{Q}_ℓ be a subset of \mathcal{Q} consisting of those cubes that meet the interior of G . Note that

$$|Y_\ell| \leq |\mathcal{Q}_\ell|. \quad (4)$$

We further split $\mathcal{Q}_\ell = \mathcal{Q}_\ell^\partial \cup \mathcal{Q}_\ell^{\text{int}}$ into two disjoint sets where $\mathcal{Q}_\ell^\partial$ contains those cubes that meet the boundary of G and $\mathcal{Q}_\ell^{\text{int}}$ contains those that are fully contained in the interior of G . See Figure 10.

We have an obvious bound on the size of $\mathcal{Q}_\ell^{\text{int}}$

$$|\mathcal{Q}_\ell^{\text{int}}| \leq \frac{1}{\varepsilon^d} \text{Vol}(G) \leq \varepsilon^{-d} \kappa^d \beta_d. \quad (5)$$

For the size of $\mathcal{Q}_\ell^\partial$ we can get the following bound. Each cube of $\mathcal{Q}_\ell^\partial$ belongs to the $(\varepsilon \sqrt{d})$ -neighborhood N_ε of the boundary ∂G of G . The $(d-1)$ -dimensional volume of ∂G can be

bounded by some function $f(d)$ depending only on d (note that G was obtained by cutting B^d at most d -times). Therefore

$$\lim_{\varepsilon \rightarrow 0} \text{Vol}(N_\varepsilon) = 0 \quad (6)$$

considering d fixed. In addition

$$|\mathcal{Q}_\ell| \leq \frac{1}{\varepsilon^d} \text{Vol}(N_\varepsilon). \quad (7)$$

Combining $|X_\ell| = n$ with (3), (4) (5), and (7) yields

$$\frac{|Y_\ell|}{|X_\ell|} \leq \frac{|\mathcal{Q}_\ell|}{n} \leq \frac{\varepsilon^{-d} \kappa^d \beta_d + \varepsilon^{-d} \text{Vol}(N_\varepsilon)}{\varepsilon^{-d} \beta_d} = \kappa^d + \frac{\text{Vol}(N_\varepsilon)}{\beta_d}.$$

This is a contradiction with $\frac{|Y_\ell|}{|X_\ell|} > \kappa^d + \frac{\zeta}{2}$ if ε is small enough using (6). \square

5 Lower bound

In this section we prove Theorem 2(2). We reuse many steps from Pach's original proof [Pac98] and we also follow an exposition of Pach's proof by Matoušek [Mat02, Chapter 9].

Lemma 13 (Few separations). *Let S_1, \dots, S_{d+1} be disjoint finite sets of points in \mathbb{R}^d and \mathbf{p} a point in \mathbb{R}^d such that $S_1 \cup S_2 \cup \dots \cup S_{d+1} \cup \{\mathbf{p}\}$ is in general position. Then there exist sets $Y_1 \subseteq S_1, \dots, Y_{d+1} \subseteq S_{d+1}$ satisfying:*

1. $|Y_i| \geq \frac{1}{2^d} |S_i|$
2. the point \mathbf{p} either lies in all (Y_1, \dots, Y_{d+1}) -simplices, or in none of them.

Remark 14. It might be interesting to compare Lemma 13, and its measure version, Lemma 18 (in our appendix), with Lemma 7.4 in [FGL⁺12]. Lemma 18 essentially shows that only d separations are needed in Lemma 7.4 of [FGL⁺12] (the assumptions on the measures in Lemma 18 are slightly stronger when compared with Lemma 7.4 of [FGL⁺12], but this does not mean any difference in the proof). That is, the constant in Lemma 7.4 of [FGL⁺12] can be improved from $2^{2-2^{d+1}}$ to $2^{-d} - \delta$ for arbitrarily small $\delta > 0$.

Proof. We will reduce the sizes of the sets S_i in $d+1$ steps, after these steps we obtain the required Y_i . We set $S_i^{(0)} := S_i$ for $i \in [d+1]$ and in the j th step we aim to construct sets $S_i^{(j)}$ for $i, j \in [d+1]$ and hyperplanes H'_j with the following properties.

- (i) $S_i^{(j)} \subseteq S_i^{(j-1)}$ for $i, j \in [d+1]$;
- (ii) $|S_i^{(j)}| \geq |S_i^{(j-1)}|/2$ for $i, j \in [d+1]$, $i \neq j$;
- (iii) $S_j^{(j)} = S_j^{(j-1)}$ for $j \in [d+1]$; and
- (iv) H'_j strictly separates \mathbf{p} from $S_i^{(j)}$ for $i, j \in [d+1]$, $i \neq j$.

This can be easily done inductively using the Ham-Sandwich theorem. In the j th step we assume that we have already constructed the sets $S_i^{(j')}$ and the hyperplanes $H'_{j'}$, for $j' < j$. By the general position version of Ham-Sandwich theorem [Mat03, Corollary 3.1.3] there is a hyperplane H''_j simultaneously bisecting the d sets $S_i^{(j-1)}$ for $i \neq j$. That is, both open half

spaces determined by H_j'' contain at least $\lfloor |S_i^{(j-1)}|/2 \rfloor$ points of each $S_i^{(j-1)}$ for $i \in [d+1] \setminus \{j\}$. We would like to choose $S_i^{(j)}$ to be the half of $S_i^{(j-1)}$ which belongs to the opposite halfspace than \mathbf{p} obtaining the required conclusion.

We just have to be careful enough when \mathbf{p} actually belongs to H_j'' or when H_j'' intersects some $S_i^{(j-1)}$ for $i \in [d+1] \setminus \{j\}$. If $\mathbf{p} \in H_j''$, we consider the (possibly empty) set U of those points of H_j'' which simultaneously belong to $S_1 \cup \dots \cup S_{d+1}$. We realize that the flat determined by U (i.e., the affine hull of U) is strictly contained in H_j'' and \mathbf{p} does not belong to this flat, both by the general position assumption on $\{\mathbf{p}\} \cup U$. Therefore, we can perturb H_j'' a bit so that it still contains U but it avoids \mathbf{p} and no other point of $S_1 \cup \dots \cup S_{d+1}$ switched the side. Therefore we can assume that \mathbf{p} does not belong to H_j'' .

As soon as we know that \mathbf{p} does not belong to H_j'' we consider the hyperplane H_j' obtained by shifting H_j'' a small bit towards \mathbf{p} . For $i \in [d+1] \setminus j$ we set $S_i^{(j)}$ to be the subset of $S_i^{(j-1)}$ belonging to the open halfspace on the other side of H_j' than \mathbf{p} . We also set $S_j^{(j)} := S_j^{(j-1)}$. Then these sets satisfy the required conditions (i)–(iv).

Finally, we set $Y_i := S_i^{(d+1)}$ for $i \in [d+1]$. Then $Y_i \subseteq S_i$ and $|Y_i| \geq \frac{1}{2^d} |S_i|$ by (i), (ii) and (iii). We slightly perturb the hyperplanes H_j' obtaining new hyperplanes H_j in general position such that each H_j still strictly separates \mathbf{p} and Y_i . Letting \mathcal{H} to be the arrangement of these hyperplanes we get either $\mathbf{p} \in \Delta(\mathcal{H})$ or not.

In the first case Lemma 7 and Lemma 9 imply that \mathbf{p} is in all (Y_1, \dots, Y_{d+1}) -simplices. In the second case Lemma 7 implies that \mathbf{p} is in no (Y_1, \dots, Y_{d+1}) -simplex. \square

The last tool we need for the proof of Theorem 1 is the weak hypergraph regularity lemma. We will be given k -partite k -uniform hypergraph \mathbf{H} on vertex set $X_1 \cup \dots \cup X_k$. That is, each edge of the hypergraph contains exactly one point from each of the X_i (assuming that X_i are pairwise disjoint). Given subsets $Y_i \subseteq X_i$ for $i \in [k]$ we define $e(Y_1, \dots, Y_k)$ as the number of edges in the subhypergraph $\mathbf{H}[Y_1, \dots, Y_k]$ induced by Y_1, \dots, Y_k and the *density* function

$$\rho(Y_1, \dots, Y_k) := \frac{e(Y_1, \dots, Y_k)}{|Y_1| \cdots |Y_k|}$$

as the ratio of the number of edges in $\mathbf{H}[Y_1, \dots, Y_k]$ and the number of all possible edges in a k -partite hypergraph with vertex set $Y_1 \cup \dots \cup Y_k$. We also set $\rho(\mathbf{H}) := \rho(X_1, \dots, X_k)$.

Theorem 15 (Weak regularity lemma for hypergraphs [Pac98]; see also [Mat02, Theorem 9.4.1]). *Let \mathbf{H} be a k -partite k -uniform hypergraph on a vertex set $X_1 \cup \dots \cup X_k$, where $|X_i| = n$ for $i \in [k]$. Suppose that its edge density satisfy $\rho(\mathbf{H}) \geq \beta$ for some $\beta > 0$. Let $0 < \varepsilon < \frac{1}{2}$. Suppose also that n is sufficiently large in terms of k, ε and β .*

Then there exist subsets $S_i \subseteq X_i$ of equal size $|S_i| = s \geq \beta^{1/\varepsilon^k} n$, for any $i \in [k]$ such that

1. (High density) $\rho(S_1, \dots, S_k) \geq \beta$, and
2. (Edges on large subsets) $e(Y_1, \dots, Y_k) > 0$ for any $Y_i \subseteq S_i$ with $|Y_i| \geq \varepsilon s, i = 1, 2, \dots, k$.

We are finally ready to prove the first main result. That is we prove that Pach's constant from Theorem 1 is at most $2^{-2^{d^2+3d}}$.

Proof of Theorem 2(2). It is convenient to start the proof with additional assumptions. Later on we will show how to remove these assumptions. We start assuming that $X_1 \cup \dots \cup X_{d+1}$ is in general position and also assuming that size n of the sets X_i is large enough, that is, $n \geq n_0$, where n_0 depends only on d .

By Theorem 4, there is a point \mathbf{p} contained in the interior of at least $\frac{1}{(d+1)!}n^{d+1} - O(n^d)$ (X_1, \dots, X_{d+1}) -simplices. We perturb the point \mathbf{p} a little so that $X_1 \cup \dots \cup X_{d+1} \cup \{\mathbf{p}\}$ is in general position (and it does not leave interior of any (X_1, \dots, X_{d+1}) -simplex during the perturbation). We require that n_0 is large enough so that \mathbf{p} actually belongs to the interior of at least $\frac{1}{2^{d^2}}n^{d+1}$ (X_1, \dots, X_{d+1}) -simplices using a very rough estimate $(d+1)! < 2^{d^2}$ (a better estimate would not improve the bound significantly).

Next, we consider $(d+1)$ -partite hypergraph \mathbf{H} with vertex set $X_1 \cup X_2 \cup \dots \cup X_{d+1}$, where edges are precisely the (X_1, \dots, X_{d+1}) -simplices containing the point \mathbf{p} . Let $\varepsilon = \frac{1}{2^d}$ and let us further require that n_0 is large enough so that the assumptions of Theorem 15 are met. We apply the weak regularity lemma (Theorem 15) to \mathbf{H} . Note that $\beta \geq \frac{1}{2^{d^2}}$. This yields sets $S_i \subseteq X_i$ with size $|S_i| = s \geq \beta^{1/\varepsilon^d}n$, and such that any subsets $Y_i \subseteq S_i$ of size at least εs induce an edge; that is, there is a (Y_1, \dots, Y_{d+1}) -simplex containing the point \mathbf{p} .

Finally, we apply Lemma 13 with the sets S_1, \dots, S_{d+1} and point \mathbf{p} . We obtain sets $Y_i \subseteq S_i$ with $|Y_i| \geq \frac{1}{2^d}s = \varepsilon s$. Moreover, the point \mathbf{p} either lies in all (Y_1, \dots, Y_{d+1}) -simplices, or in none of them. But the latter possibility is excluded by the fact that Y_i are large enough.

Because $c_1^{sup} = 1/2$, we assume $d \geq 2$ in the following calculations. So we obtained the desired sets Y_i 's of size $c_d|X_i|$, where

$$c_d \geq \frac{1}{2^d} \beta^{1/\varepsilon^{d+1}} \geq \frac{1}{2^d} \cdot \left(\frac{1}{2^{d^2}} \right)^{2^{d(d+1)}} = 2^{-d-d^2 \cdot 2^{d(d+1)}} \geq 2^{-2^{d^2+3d}}.$$

This finishes the proof under the assumptions that $X_1 \cup \dots \cup X_{d+1}$ is in general position and $n \geq n_0$.

First, by a standard compactness argument we can remove the general position assumption. Here we can even assume that X_i are multisets, that is, some of the points can repeat more than once. Indeed, we choose sets $X_i^{(n)}$ such that $X_1^{(n)} \cup \dots \cup X_d^{(n)}$ is in general position for every positive integer n and such that $X_i^{(n)}$ converges to X_i . We obtain the corresponding sets $Y_i^{(n)}$ and Pach points $\mathbf{p}^{(n)}$ using the general position version of the theorem. Since X_i are finite there is an infinite increasing sequence (n_k) such that $Y_i^{(n_k)}$ converge to certain sets $Y_i \subseteq X_i$. Since all $X_i^{(n)}$ belong to a compact region in \mathbb{R}^d , the sequence of Pach points $\mathbf{p}^{(n_k)}$ has a mass point \mathbf{p} . It is routine to check that the sets Y_i and the point \mathbf{p} satisfy the required conditions.

Next, we can remove the assumption $n \geq n_0$ in the following way. If $n < n_0$ we find an integer m such that $m \cdot n \geq n_0$. We make multisets X'_i where X'_i consist of points of X_i , each repeated m -times. We find Y'_i of sizes at least $c_d \cdot m \cdot n$ and \mathbf{p}' for X'_i . Forgetting the m -fold repetitions in Y'_i we get the required Y_i of sizes at least $|Y'_i|/m$ with $\mathbf{p} = \mathbf{p}'$. \square

Remark 16. The argument at the end of the previous proof also shows that the assumption that all X_i have equal size can be easily removed. Indeed, let X_1, \dots, X_{d+1} be subsets of \mathbb{R}^d of various sizes. We set $\gamma := |X_1| \cdots |X_{d+1}|$. We create multisets X'_i where each point of X_i repeats $\gamma/|X_i|$ times. That is, each X'_i has size γ and we can find Y'_i of sizes at least $c_d \gamma$ and \mathbf{p}' for X'_i . Forgetting the repetitions in Y'_i we get Y_i of sizes at least $c_d|X_i|$.

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References

- [Bár82] I. Bárány. A generalization of Carathéodory’s theorem. *Discrete Math.*, 40:141–152, 1982.
- [BF84] E. Boros and Z. Füredi. The number of triangles covering the center of an n -set. *Geom. Dedicata*, 17(1):69–77, 1984.
- [BP14] I. Bárány and J. Pach. Homogeneous selections from hyperplanes. *J. Combin. Theory Ser. B*, 104:81–87, 2014.
- [FGL⁺12] J. Fox, M. Gromov, V. Lafforgue, A. Naor, and J. Pach. Overlap properties of geometric expanders. *J. Reine Angew. Math.*, 671:49–83, 2012.
- [Gro10] M. Gromov. Singularities, expanders and topology of maps. Part 2: From combinatorics to topology via algebraic isoperimetry. *Geom. Funct. Anal.*, 20(2):416–526, 2010.
- [Kar12] R. Karasev. A simpler proof of the Boros-Füredi-Bárány-Pach-Gromov theorem. *Discrete Comput. Geom.*, 47(3):492–495, 2012.
- [Kla04] D. A. Klain. The Minkowski problem for polytopes. *Adv. Math.*, 185(2):270–288, 2004.
- [KMS12] D. Král’, L. Mach, and J.-S. Sereni. A new lower bound based on Gromov’s method of selecting heavily covered points. *Discrete Comput. Geom.*, 48(2):487–498, 2012.
- [Mat02] J. Matoušek. *Lectures on Discrete Geometry*. Springer-Verlag New York, Inc., 2002.
- [Mat03] J. Matoušek. *Using the Borsuk-Ulam Theorem*. Springer, Berlin etc., 2003.
- [MW11] J. Matoušek and U. Wagner. On Gromov’s method of selecting heavily covered points. Preprint; <http://arxiv.org/abs/1102.3515>, 2011.
- [Pac98] J. Pach. A Tverberg-type result on multicolored simplices. *Comput. Geom.*, 10:71–76, 1998.
- [Räc06] H. Räcke. Measure concentration for the sphere. Lecture notes, <http://www.dcs.warwick.ac.uk/~harry/teaching/pdf/lecture13.pdf>, 2006.

A Measure version of Pach's theorem

Here we prove Theorem 5.

We recall that by the absolute continuity of a measure μ we mean that $\mu(A) = 0$ whenever the Lebesgue measure of A is 0 and also that the σ -algebra of μ -measurable sets coincides with the σ -algebra of Lebesgue-measurable sets. An important property of absolutely continuous measures that we will use several times is that whenever A is a measurable set with $\mu(A) > 0$ and $r \in (0, \mu(A))$, then there is a subset R of A such that $\mu(R) = r$.³ As we mentioned in the introduction, we need a measure-theoretic analogue of the weak hypergraph regularity lemma.

Theorem 17 (Weak regularity lemma for measures). *Let μ_1, \dots, μ_k be absolutely continuous probability measures on \mathbb{R}^d . Let $\mu := \mu_1 \times \dots \times \mu_k$ be the product measure and let $E \subseteq \prod_{i=1}^k \mathbb{R}^d$ be a measurable set with $\mu(E) > \beta > 0$. Then for every $\varepsilon \in (0, \frac{1}{2})$ there exist measurable sets $S_i \subseteq \mathbb{R}^d$ satisfying the following properties.⁴*

1. $\mu_i(S_i) = s$ for some $s \geq \beta^{1/\varepsilon^k}$ and every $i \in [k]$;
2. $\mu(E \cap \prod_{i=1}^k S_i) \geq \beta s^k$; and
3. $\mu((Y_1 \times \dots \times Y_k) \cap E) > 0$ for any measurable sets $Y_i \subseteq S_i$ with $\mu_i(Y_i) \geq \varepsilon s$.

We first prove Theorem 5 assuming Theorem 17. We prove Theorem 17 at the very end of the appendix.

We need the following analogue of Lemma 13.

Lemma 18. *Let μ_1, \dots, μ_{d+1} be absolutely continuous probability measures in \mathbb{R}^d , S_1, \dots, S_{d+1} be measurable subsets of \mathbb{R}^d , and $\mathbf{p} \in \mathbb{R}^d$ be a point. Then for every $\delta > 0$, there exist sets $Y_1 \subseteq S_1, \dots, Y_{d+1} \subseteq S_{d+1}$ satisfying $\mu_i(Y_i) \geq (\frac{1}{2^d} - \delta) \mu_i(S_i)$ such that either \mathbf{p} lies in all (Y_1, \dots, Y_{d+1}) -simplices or in none of them.*

The proof of this lemma is a direct analogue of the proof of Lemma 13. We include it for completeness.

Proof. We set $S_i^{(0)} := S_i$ for $i \in [d+1]$ and in the j th step we aim to construct sets $S_i^{(j)}$ for $i, j \in [d+1]$ and hyperplanes H_j' with the following properties.

- (i) $S_i^{(j)} \subseteq S_i^{(j-1)}$ for $i, j \in [d+1]$;
- (ii) $\mu_i(S_i^{(j)}) \geq \mu_i(S_i^{(j-1)})/2$ for $i, j \in [d+1]$, $i \neq j$;
- (iii) $S_j^{(j)} = S_j^{(j-1)}$ for $j \in [d+1]$; and

³In fact, by checking the proof, this weaker condition is sufficient for the proof of the weak regularity lemma for measures below (Theorem 17).

⁴It may be helpful to see how Theorem 17 relates to its discrete counterpart. We forget that the measures have to be absolutely continuous for a while. Let us start with sets X_i . We replace the set X_i with probability measure μ_i defined as follows if $x \in X_i$, then $\mu_i(x) = 1/|X_i|$, otherwise $\mu_i(x) = 0$. The measure μ then represents a choice of a random (X_1, \dots, X_{d+1}) -simplex. The set E corresponds to the hypergraph \mathbf{H} in Theorem 15. The measure $\mu(E)$ then corresponds to the density $\rho(\mathbf{H})$. For a technical reason, in Theorem 17 we assume slightly stronger condition $\mu(E) > \beta$ when compared with $\rho(\mathbf{H}) \geq \beta$ in Theorem 15.

(iv) H'_j weakly separates⁵ \mathbf{p} from $S_i^{(j)}$ for $i, j \in [d+1]$, $i \neq j$.

In the j th step we assume that we have already constructed the sets $S_i^{(j')}$ and the hyperplanes $H'_{j'}$ for $j' < j$. By the measure version of Ham-Sandwich theorem [Mat03, Theorem 3.1.1] there is a hyperplane H'_j simultaneously bisecting the d sets $S_i^{(j-1)}$ for $i \neq j$. That is, both open half spaces determined by H'_j have measure $\mu_i(S_i^{(j-1)})/2$ for each $i \in [d+1] \setminus \{j\}$. We choose $S_i^{(j)}$ to be the half of $S_i^{(j-1)}$ which belongs to the opposite halfspace than \mathbf{p} obtaining the required conclusion (if $\mathbf{p} \in H'_j$, we choose any of the two halves).

We set $Y'_i := S_i^{(d+1)}$. Then $\mu_i(Y'_i) \geq 1/2^d \mu_i(S_i)$. We perturb the hyperplanes slightly and adjust the sets Y'_i accordingly obtaining hyperplanes H_1, \dots, H_{d+1} and sets Y_1, \dots, Y_{d+1} such that $\mathcal{H} := (H_1, \dots, H_{d+1})$ is in general position, $\mu_i(Y_i) \geq (1/2^d - \delta)\mu_i(S_i)$ and H_j strictly separates \mathbf{p} and Y_i for any $i, j \in [d+1]$, $i \neq j$.

Now we get either $\mathbf{p} \in \Delta(\mathcal{H})$ or not. In the first case Lemma 7 and Lemma 9 imply that \mathbf{p} is in all (Y_1, \dots, Y_{d+1}) -simplices. In the second case Lemma 7 implies that \mathbf{p} is in no (Y_1, \dots, Y_{d+1}) -simplex. \square

Proof of Theorem 5. Let $\mu = \mu_1 \times \dots \times \mu_{d+1}$ be the product measure. By the main result of [Kar12] there is a point \mathbf{p} such that the probability that a random μ -simplex contains \mathbf{p} is at least $\frac{1}{(d+1)!}$.

We let E be the set of all points $(\mathbf{x}_1, \dots, \mathbf{x}_{d+1}) \in (\mathbb{R}^d)^{d+1}$ such that the simplex formed by $\mathbf{x}_1, \dots, \mathbf{x}_{d+1}$ (possibly degenerate) contains P . We let $\delta > 0$ to be a small enough parameter, in particular $\delta < \frac{1}{(d+1)!} - \frac{1}{2^{d^2}}$. We will now use the hypergraph regularity lemma (Theorem 17), with $k = d+1$, $\beta = \frac{1}{2^{d^2}}$, $\varepsilon = \frac{1}{2^d} - \delta$, and E as we set it above. We obtain sets S_1, \dots, S_{d+1} with $\mu_i(S_i) \geq \beta \frac{1}{\varepsilon^{d+1}}$.

Let Y_1, \dots, Y_{d+1} be the sets obtained from S_1, \dots, S_{d+1} and \mathbf{p} using Lemma 18. The point \mathbf{p} either lies in all (Y_1, \dots, Y_{d+1}) -simplices, or in none of them. Since $\mu_i(Y_i) \geq \varepsilon \mu_i(S_i)$, we deduce that $\mu((Y_1 \times \dots \times Y_{d+1}) \cap E) > 0$, and, therefore, the latter possibility does not happen.

It remains to calculate the measures of sets Y_i .

$$\mu_i(Y_i) \geq \varepsilon \mu(S_i) \geq \left(\frac{1}{2^d} - \delta\right) \left(\frac{1}{2^{d^2}}\right)^{\frac{1}{\left(\frac{1}{2^d} - \delta\right)^{d+1}}} \geq 2^{-2^{d^2} + 3d}.$$

The last inequality is valid for sufficiently small $\delta > 0$ by a similar computation as in the proof of Theorem 2(2). \square

It remains to prove Theorem 17. In the proof we follow the proof of the discrete version by Pach [Pac98] and the exposition by Matoušek [Mat02] (slightly improving the constant in the theorem). It turns out that some parts of the proof can be taken almost in verbatim. We need, however, the following measure-theoretic replacement of the averaging argument in the discrete version.

Lemma 19. *Let μ be an absolutely continuous measure in \mathbb{R}^d , h be non-negative measurable real function in \mathbb{R}^d and $A \subseteq \mathbb{R}^d$ a measurable set. Let $a := \mu(A) \in \mathbb{R}$, $t := \int_A h d\mu$, and let $r \in (0, a)$. Then, there exists a set $R \subseteq A$ with $\mu(R) = r$ and $\int_R h d\mu \geq \frac{r}{a} \cdot t$.*

⁵That is, \mathbf{p} fits into one of the *closed* subspaces determined by H'_j and $S_i^{(j)}$ belongs to the second one.

Proof. For a given real parameter p let us consider sets $R_p^> := \{x \in A : h(x) > p\}$ and $R_p^\geq := \{x \in A : h(x) \geq p\}$. Let $p_0 := \sup\{p : \mu(R_p^\geq) \geq r\}$. Note that p_0 is finite. Indeed, if $\mu(R_p^\geq) \geq r$, then $t \geq \int_{R_p^\geq} h d\mu \geq \int_{R_p^\geq} p d\mu \geq rp$ concluding $p \leq t/r$. We also deduce that $\mu(R_{p_0}^>) \leq r$ and $\mu(R_{p_0}^\geq) \geq r$ (both of these claims follow from the continuity of measure, considering the sequence $(R_{p_0+1/i}^\geq)_{i=1}^\infty$ in the first case and $(R_{p_0-1/i}^\geq)_{i=1}^\infty$ in the second case). Therefore there is a measurable set R with $R_{p_0}^> \subseteq R \subseteq R_{p_0}^\geq$ such that $\mu(R) = r$. The choice of R implies that $h(x) \geq p_0 \geq h(y)$ for any $x \in R$ and $y \in A \setminus R$. Therefore

$$\int_R h d\mu \geq \frac{r}{a} \int_A h d\mu$$

as required. \square

Proof of Theorem 17. Let $\zeta := \mu(E) - \beta$. To simplify our notation let $\mu_E(S_1 \times \cdots \times S_k)$ stand for $\mu((S_1 \times \cdots \times S_k) \cap E)$. Intuitively, we want to choose S_i as not too small sets such that $\mu_E(S_1 \times \cdots \times S_k)$ is as big as possible. This choice can be guided by *magical* density defined as

$$\rho(S_1, \dots, S_k) = \frac{\mu_E(S_1 \times \cdots \times S_k)}{(\mu_1(S_1) \cdots \mu_k(S_k))^{1-\varepsilon^k/k}}.$$

Let c be the supremum of $\rho(S_1, \dots, S_k)$ taken over all measurable subsets $S_i \subseteq \mathbb{R}^d$ with $\mu_1(S_1) = \mu_2(S_2) = \cdots = \mu_k(S_k)$. First we consider any measurable sets $S_1, \dots, S_k \subseteq \mathbb{R}^d$ with $\mu_1(S_1) = \cdots = \mu_k(S_k)$ satisfying $\rho(S_1, \dots, S_k) \geq c - \zeta$. We verify that any collection of such sets satisfy conditions (1) and (2).

Let $s := \mu_1(S_1) = \cdots = \mu_k(S_k)$. Then

$$\begin{aligned} \mu_E(S_1 \times \cdots \times S_k) &= \rho(S_1, \dots, S_k) s^{k-\varepsilon^k} \geq \left(\rho(\mathbb{R}^d, \dots, \mathbb{R}^d) - \zeta \right) s^{k-\varepsilon^k} \\ &= (\mu(E) - \zeta) s^{k-\varepsilon^k} = \beta s^{k-\varepsilon^k} \geq \beta s^k, \end{aligned}$$

obtaining (2).

Since $\mu_E(S_1 \times \cdots \times S_k) \leq \mu(S_1 \times \cdots \times S_k) = s^k$, it follows that $\rho(S_1, \dots, S_k) \leq s^{\varepsilon^k}$. On the other hand, $\rho(S_1, \dots, S_k) \geq \beta$ as derived when checking (2). Combining these two bounds provides (1).

Now let S'_1, \dots, S'_k be any measurable sets in \mathbb{R}^d such that $\mu_1(S'_1) = \cdots = \mu_k(S'_k) =: s'$ and $s' \leq (1-\varepsilon)s$. By the choice of S_1, \dots, S_k we know that $\rho(S_1, \dots, S_k) \geq \rho(S'_1, \dots, S'_k) - \zeta$. However, the subsequent computations, for verifying (3) will be much easier, if we can achieve a stronger inequality $\rho(S_1, \dots, S_k) \geq \rho(S'_1, \dots, S'_k)$.

This stronger inequality can be indeed achieved. If there are S'_1, \dots, S'_k such that $s' \leq (1-\varepsilon)s$ and $\rho(S_1, \dots, S_k) < \rho(S'_1, \dots, S'_k)$, then we simply replace S_1, \dots, S_k with S'_1, \dots, S'_k and ‘restart’. These replacements have to finish in a finite number of steps because by each replacement we decrease the value of s at least by a factor of $(1-\varepsilon)$ whereas the value of s cannot be below β^{1/ε^k} due to (1).

Now we can proceed with checking (3). Let $Y_i \subseteq S_i$ be sets with $\mu_i(Y_i) = \varepsilon s$ (if $\mu_i(Y_i) > \varepsilon s$

we can consider slightly smaller sets because we assume absolute continuity). We have

$$\begin{aligned}
\mu_E(Y_1 \times \cdots \times Y_k) &= \mu_E(S_1 \times \cdots \times S_k) \\
&\quad - \mu_E((S_1 \setminus Y_1) \times S_2 \times S_3 \times \cdots \times S_k) \\
&\quad - \mu_E(Y_1 \times (S_2 \setminus Y_2) \times S_3 \times \cdots \times S_k) \\
&\quad \vdots \\
&\quad - \mu_E(Y_1 \times Y_2 \times Y_3 \times \cdots \times (S_k \setminus Y_k)).
\end{aligned}$$

By Fubini's theorem, $\mu_E(S_1 \times S_2 \times \cdots \times S_k)$ equals $\int_{S_1} h d\mu_1$, for some non-negative measurable function h . Therefore, by Lemma 19, for every $r \in (0; \mu_1(S_1))$, there exists $R_1 \subseteq S_1$ with $\mu_1(R_1) = r$ and $\mu_E(R_1 \times S_2 \times \cdots \times S_k) \geq \frac{r}{\mu_1(S_1)} \mu_E(S_1 \times S_2 \times \cdots \times S_k)$.

Using this observation repeatedly at different coordinates and setting $r := \mu_1(S_1 \setminus Y_1) = (1 - \varepsilon)s$, we obtain sets $R_2 \subseteq S_2, \dots, R_k \subseteq S_k$ with $\mu_i(R_i) = r$ for $i \geq 2$ such that

$$\mu_E((S_1 \setminus Y_1) \times R_2 \times \cdots \times R_k) \geq \left(\frac{r}{s}\right)^{k-1} \mu_E((S_1 \setminus Y_1) \times S_2 \times \cdots \times S_k).$$

Therefore

$$\begin{aligned}
\mu_E((S_1 \setminus Y_1) \times S_2 \times \cdots \times S_k) &\leq \left(\frac{s}{r}\right)^{k-1} \mu_E((S_1 \setminus Y_1) \times R_2 \times \cdots \times R_k) \\
&= \left(\frac{s}{r}\right)^{k-1} r^{k-\varepsilon^k} \rho(S_1 \setminus Y_1, R_2, \dots, R_k) \\
&\leq \left(\frac{s}{r}\right)^{k-1} r^{k-\varepsilon^k} \rho(S_1, S_2, \dots, S_k) \\
&= \left(\frac{s}{r}\right)^{k-1} \left(\frac{r}{s}\right)^{k-\varepsilon^k} \mu_E(S_1 \times S_2 \times \cdots \times S_k) \\
&= (1 - \varepsilon)^{1-\varepsilon^k} \mu_E(S_1 \times S_2 \times \cdots \times S_k).
\end{aligned}$$

Note that in the inequality in the third line we have used that $r \leq (1 - \varepsilon)s$ implying we indeed can use the stronger inequality for comparing the magical densities as announced above. Finally, we deduce

$$\mu_E((S_1 \setminus Y_1) \times S_2 \times \cdots \times S_k) \leq (1 - \varepsilon) \mu_E(S_1 \times S_2 \times \cdots \times S_k).$$

For $i \geq 2$ we can get a bound

$$\mu_E(Y_1 \times \cdots \times Y_{i-1} \times S_i \setminus Y_i \times S_{i+1} \times \cdots \times S_k) \leq (1 - \varepsilon) \varepsilon^{i-1-\varepsilon^k} \mu_E(S_1 \times S_2 \times \cdots \times S_k)$$

by a similar computation; this time for $r = \varepsilon s$ and sets $R_i \subseteq S_i \setminus Y_i, R_{i+1} \subseteq S_{i+1}, \dots, R_k \subseteq S_k$.

Altogether $\mu_E(Y_1 \times \cdots \times Y_k)$ is at least $1 - (1 - \varepsilon) - (1 - \varepsilon) \sum_{i=2}^k \varepsilon^{i-1-\varepsilon^k}$ factor of $\mu_E(S_1 \times \cdots \times S_k)$, and it can be checked by an elementary computation that this factor is strictly positive. (This is checked in [Mat02] at the end of the proof of Theorem 9.4.1 and we do not repeat this computation here.) \square